Supplemental Summer Calculus Packet (for courses that don't have a school summer packet)

Graphically estimating limits

Statement of problem

1. ☐ Check that problem essentially asks, "Find the limit, if possible".
2. ☐ Copy the function and requested limit of form $\lim_{x\to c} f(x)$ in your handwriting.
3. Set aside a 4" × 4" workspace for drawing a huge graph of the function:
   a. ☐ Draw a tall y-axis with an upward arrowhead labeled y.
   b. ☐ Draw a wide x-axis with a rightward arrowhead labeled x.
   c. ☐ Label the origin O.
   d. ☐ Draw x-ticks for important x-values, if any.
   e. ☐ Draw y-ticks for important y-values, if any.
   f. ☐ Draw the graph.
   g. ☐ Use arrowheads at ends of viewable portions of graph where graph extends forever beyond the viewing area.

Tool for analyzing situation

4. ☐ Say "If making x become arbitrarily close to c from the left & the right makes $f(x)$ become arbitrarily close to L, we say $\lim_{x\to c} f(x) = L$."

Equivalent parts in Statement and Tool

5. For each substep below, slash-replace the letter c and attempt the step. If a step can't be sensibly done, say the analysis ends (skip to 8).
   a. ☐ On the x-axis, thicken and label the x-tick for $x = c$.
   b. ☐ On the x-axis, draw an x-tick labeled x_1 slightly to the left of the x-tick labeled c. Then draw a 2nd x-tick labeled x_2 between the x-ticks labeled x_1 and c.
   c. ☐ Near the x-axis, draw a small horizontal arrow starting from slightly left of the x-tick labeled x_1, passing by the x-ticks labeled x_1 and x_2, and heading toward the x-tick labeled c.
   d. ☐ On the x-axis, draw an x-tick labeled x_3 slightly to the right of the x-tick labeled c. Then draw a 2nd x-tick labeled x_4 between the x-ticks labeled x_3 and c.
   e. ☐ Near the x-axis, draw a small horizontal arrow starting from slightly right of the x-tick labeled x_3, passing by the x-ticks labeled x_3 and x_4, and heading toward the x-tick labeled c.
   f. ☐ Say, "x becomes arbitrarily close to c from the left & the right."
6. For each substep below, slash-replace the expression $f(x)$ and attempt the step. If a step can't be sensibly done, say the analysis ends (skip to 8).
   a. ☐ On the x-axis, find the x-ticks for x_1 and x_2. On the graph of $y = f(x)$, draw the corresponding points, labeled P_1 and P_2, respectively.
   b. ☐ Near the graph of $y = f(x)$, draw a small graph-hugging arrow riding along the graph and passing by P_1 and then P_2, in that order.
   c. ☐ On the x-axis, find the x-ticks for x_3 and x_4. On the graph of $y = f(x)$, draw the corresponding points, labeled P_3 and P_4, respectively.
   d. ☐ Near the graph of $y = f(x)$, draw a small graph-hugging arrow riding along the graph and passing by P_3 and P_4, in that order.
7. For each substep below, slash-replace the expression $f(x)$ and the letter L and attempt the step. If a step can't be sensibly done, say the analysis ends (skip to 8).
   a. ☐ On the graph of $y = f(x)$, find P_1 and P_2. On the y-axis, draw the corresponding y-ticks, labeled y_1 and y_2 respectively.
   b. ☐ Near the y-axis, draw a small vertical arrow (maybe too small to see) passing by the y-ticks labeled y_1 and y_2, in that order.
   c. ☐ On the graph of $y = f(x)$, find P_3 and P_4. On the y-axis, draw the corresponding y-ticks, labeled y_3 and y_4 respectively.
   d. ☐ Near the y-axis, draw a small vertical arrow (maybe too small to see) passing by the y-ticks labeled y_3 and y_4, in that order.
   e. ☐ On the y-axis, thicken a y-tick labeled L where the two vertical arrows near the y-axis seem to meet.
   f. ☐ Take a moment to look at the vertical arrows near the y-axis and the y-tick labeled L. Say, "$f(x)$ is thus caused to become arbitrarily close to L."
8. Perform quality control:
   a. ☐ Look at the x-ticks and arrows from step 5.
   b. ☐ Look at the points and graph-hugging arrows from step 6.
   c. ☐ Look at the vertical arrows and y-tick from step 7.
   d. ☐ Would trying to redraw one or more x-ticks from step 5 change the result of the search for the y-tick in step 7? If so, start over from step 5.

Populated tool

9. Slash-replace the expressions c, $f(x)$, and L in the substep below.
   a. ☐ Say "Making x become arbitrarily close to c from the left & the right indeed makes $f(x)$ become arbitrarily close to L, so we say $\lim_{x\to c} f(x) = L$

Exercises: Graphically estimating limits

Problem A. On separate pages (2-4 problems per page), graphically estimate each limit below. Organize your results in the "Result (value)" column of Table A.

Table A.

	Shiftable inputs	**Theoretical Expectations**						**Actual/Measured**
		Proposition A	Proposition B	Proposition C	Proposition D	Proposition E	Proposition F	
#	Problem							Result (value)
1.	$\lim_{x\to 2} 5$							
2.	$\lim_{x\to -3} 2$							
3.	$\lim_{x\to 0} 1$							
4.	$\lim_{x\to -5} x$							
5.	$\lim_{x\to 2} x$							
6.	$\lim_{x\to 10} x$							
7.	$\lim_{x\to 2} x^2$							
8.	$\lim_{x\to 2} \left[\frac{1}{2}x^2\right]$							
9.	$\lim_{x\to -3} (x+1)$							
10.	$\lim_{x\to -3} [2(x+1)]$							

Supplemental Summer Calculus Packet (for courses that don't have a school summer packet)

Problem B. On separate pages (2-4 problems per page), graphically estimate each limit below. Organize your results in the "Result (value)" column of Table B.

Table B.

Shiftable inputs		Theoretical Expectations						Actual/Measured
		Proposition A	Proposition B	Proposition C	Proposition D	Proposition E	Proposition F	
#	Problem							Result (value)
11.	$\lim\limits_{x\to 1} \sqrt{x}$							
12.	$\lim\limits_{x\to 1} \left[2\sqrt[3]{x}\right]$							
13.	$\lim\limits_{x\to 1} \left[\sqrt{x} + 2\sqrt[3]{x}\right]$							
14.	$\lim\limits_{x\to -2} 3$							
15.	$\lim\limits_{x\to -2} x^3$							
16.	$\lim\limits_{x\to -2} [3 + x^3]$							
17.	$\lim\limits_{x\to 5} \begin{cases} -x+5, & x < 5 \\ x, & x \geq 5 \end{cases}$							
18.	$\lim\limits_{x\to 1} \begin{cases} 2, & x < 1 \\ 5, & x \geq 1 \end{cases}$							
19.	$\lim\limits_{x\to 1} \begin{cases} 2, & x < 1 \\ -1, & x \geq 1 \end{cases}$							
20.	$\lim\limits_{x\to 1} \begin{cases} 4, & x < 1 \\ 4, & x \geq 1 \end{cases}$							

Use STEAM to make plausible the laws for the limit of a constant, limit of the identity, limit of a scale multiple, and limit of a sum

Do this activity with a coach.

1. In Table A, use algebraic notation to write a proposed pattern in the blank cell under Proposition A. In the rest of the same column, write the results your proposition predicts for the ten limits in the table.
2. Check whether any entries in the Proposition A column conflict with results in the Result (value) column. Mark such conflicts 🚫. If there is even one conflict, mark Proposition A itself with a 🚫.
3. For the coach: Guide the students to propose and test propositions (use the columns for Proposition B, Proposition C, and so on) until the laws for the limit of a **constant**, the limit of the **identify** function, and the limit of a **scale multiple** are made plausible.
4. In Table B, use algebraic notation to write a proposed pattern in the blank cell under Proposition A. In the rest of the same column, write the results your proposition predicts for the ten limits in the table.
5. Check whether any entries in the Proposition A column conflict with results in the Result (value) column. Mark such conflicts 🚫. If there is even one conflict, mark Proposition A itself with a 🚫.
6. For the coach: Guide the students to propose and test propositions (use the columns for Proposition B, Proposition C, and so on) until the law for the limit of a **sum** is made plausible.

Supplemental Summer Calculus Packet (for courses that don't have a school summer packet)

Make a note card with basic limit laws

Copy these identities onto a sticky note.

1. $\left[\lim_{x\to c} k\right] = [k]$, k is a constant
2. $\left[\lim_{x\to c} x\right] = [c]$
3. $\left[\lim_{x\to c}(k[f(x)])\right] = \left[k\left(\lim_{x\to c}[f(x)]\right)\right]$, PALE (Provided All Limits Exist)
4. $\left[\lim_{x\to c}([f(x)] + [g(x)])\right] = \left[\left(\lim_{x\to c}[f(x)]\right) + \left(\lim_{x\to c}[g(x)]\right)\right]$, PALE
5. $\left[\lim_{x\to c}([f(x)] \cdot [g(x)])\right] = \left[\left(\lim_{x\to c}[f(x)]\right) \cdot \left(\lim_{x\to c}[g(x)]\right)\right]$, PALE
6. $\left[\lim_{x\to c}\left(\frac{[f(x)]}{[g(x)]}\right)\right] = \left[\frac{\left(\lim_{x\to c}[f(x)]\right)}{\left(\lim_{x\to c}[g(x)]\right)}\right]$, PALE and $\lim_{x\to c} g(x) \neq 0$
7. $\left[\lim_{x\to c}([f(x)]^n)\right] = \left[\left(\lim_{x\to c}[f(x)]\right)^n\right]$, PALE and n is a positive integer

Practice applying the basic limit laws using STEP (including tricky)

Review the **STEP** method at https://davidliao.com/literacy. On separate paper, use the **STEP** method and your note card of basic limit laws to evaluate the following limits.

1. $\lim_{x\to 2} 5x$
2. $\lim_{x\to 3}(6x^2 + 5x)$
3. $\lim_{x\to -1}[x(x^2 + 3)]$